Marshallian, hicksian and expenditure function of cobb-douglas

Suppose a consumer has the utility function $u(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2}$, where $\alpha_1, \alpha_2 > 0$ and $\alpha_1 + \alpha_2 \leq 1$. Prices are p_1, p_2 and consumer wealth is y.

1. Find the Marshallian demand functions and indirect utility function.
2. Find the expenditure function and Hicksian demand functions.

Solutions

1. The Lagrangian is:

$$L(p_1, p_2, \lambda) = x_1^{\alpha_1} x_2^{\alpha_2} - \lambda(p_1 x_1 + p_2 x_2 - y)$$

First order conditions are:

$$\begin{aligned}
\frac{\partial L}{\partial x_1} &= \alpha_1 x_1^{\alpha_1 - 1} x_2^{\alpha_2} - \lambda p_1 = 0 \to \alpha_1 x_1^{\alpha_1 - 1} x_2^{\alpha_2} = \lambda p_1 \\
\frac{\partial L}{\partial x_2} &= \alpha_2 x_1^{\alpha_1} x_2^{\alpha_2 - 1} - \lambda p_2 = 0 \to \alpha_2 x_1^{\alpha_1} x_2^{\alpha_2 - 1} = \lambda p_2 \\
\frac{\partial L}{\partial \lambda} &= p_1 x_1 + p_2 x_2 - y = 0
\end{aligned}$$

Dividing the first equation by the second, we get the optimality condition MRS = price ratio:

$$\frac{\alpha_1 x_2}{\alpha_2 x_1} = \frac{p_1}{p_2} \to p_2 x_2 = \frac{\alpha_2}{\alpha_1} p_1 x_1$$

Plugging into the budget equation, we get

$$p_1 x_1 + \frac{\alpha_2}{\alpha_1} p_1 x_1 = y \to p_1 x_1 (1 + \frac{\alpha_2}{\alpha_1}) = y$$

Therefore, the Marshallian demand function for x_1 is

$$x_1^* = \frac{\alpha_1}{\alpha_1 + \alpha_2} \frac{y}{p_1}$$

By symmetry, the solution for x_2 is

$$x_2^* = \frac{\alpha_2}{\alpha_1 + \alpha_2} \frac{y}{p_2}$$

The indirect utility function is:

$$\left(\frac{\alpha_1}{\alpha_1 + \alpha_2} \frac{y}{p_1} \right)^{\alpha_1} \left(\frac{\alpha_2}{\alpha_1 + \alpha_2} \frac{y}{p_2} \right)^{\alpha_2} = y^{\alpha_1 + \alpha_2} \left(\frac{\alpha_1}{\alpha_1 + \alpha_2} \frac{1}{p_1} \right)^{\alpha_1} \left(\frac{\alpha_2}{\alpha_1 + \alpha_2} \frac{1}{p_2} \right)^{\alpha_2}$$

2. Using the equation $u = v(p, e(p, u)))$, we get:

$$u = e(p, u)^{\alpha_1 + \alpha_2} \left(\frac{\alpha_1}{\alpha_1 + \alpha_2} \frac{1}{p_1} \right)^{\alpha_1} \left(\frac{\alpha_2}{\alpha_1 + \alpha_2} \frac{1}{p_2} \right)^{\alpha_2}$$

Solving for $e(p, u)$, we get:

$$\begin{aligned}
e(p, u) &= \left(u \left(\frac{\alpha_1}{\alpha_1 + \alpha_2} \frac{1}{p_1} \right)^{-\alpha_1} \left(\frac{\alpha_2}{\alpha_1 + \alpha_2} \frac{1}{p_2} \right)^{-\alpha_2} \right)^{\frac{1}{\alpha_1 + \alpha_2}} \\
&= u^{\frac{1}{\alpha_1 + \alpha_2}} \left(\frac{\alpha_1 + \alpha_2}{\alpha_1} \right)^{\frac{\alpha_1}{\alpha_1 + \alpha_2}} \left(\frac{\alpha_1 + \alpha_2}{\alpha_2} \right)^{\frac{\alpha_2}{\alpha_1 + \alpha_2}} p_1^{\frac{\alpha_1}{\alpha_1 + \alpha_2}} p_2^{\frac{\alpha_2}{\alpha_1 + \alpha_2}}
\end{aligned}$$

The Hicksian demand functions are given by $\frac{\partial e}{\partial p_i}$:

$$\begin{aligned}
x_{h1}(p_1, p_2, u) &= u^{\frac{1}{\alpha_1 + \alpha_2}} \left(\frac{\alpha_1 + \alpha_2}{\alpha_1} \right)^{-\frac{\alpha_2}{\alpha_1 + \alpha_2}} \left(\frac{\alpha_1 + \alpha_2}{\alpha_2} \right)^{\frac{\alpha_2}{\alpha_1 + \alpha_2}} p_1^{-\frac{\alpha_2}{\alpha_1 + \alpha_2}} p_2^{\frac{\alpha_2}{\alpha_1 + \alpha_2}} \\
x_{h2}(p_1, p_2, u) &= u^{\frac{1}{\alpha_1 + \alpha_2}} \left(\frac{\alpha_1 + \alpha_2}{\alpha_1} \right)^{\frac{\alpha_1}{\alpha_1 + \alpha_2}} \left(\frac{\alpha_1 + \alpha_2}{\alpha_2} \right)^{-\frac{\alpha_1}{\alpha_1 + \alpha_2}} p_1^{\frac{\alpha_1}{\alpha_1 + \alpha_2}} p_2^{-\frac{\alpha_1}{\alpha_1 + \alpha_2}}
\end{aligned}$$